Littlewood reliability model for modular software and Poisson approximation

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Abstract

We consider a Markovian model, proposed by Littlewood, to assess the reliability of a modular software. Specifically, we analyze the failure point process corresponding to, when reliability growth takes place. We prove the convergence in distribution of the point process to a Poisson process. Moreover, we provide a convergence rate using distance in variation. This is heavily based on a similar result of Kabanov, Liptser and Shiryaev, for a doubly-stochastic Poisson process whose intensity is driven by a Markov chain.

1 Littlewood model

Littlewood proposed in (Littlewood 1975) a Markov-type model for reliability assessment of a modular software. Basically, for a software with a finite number of modules :

- the structure of the software is represented by a finite continuous time Markov chain $X = (X_t)_{t\geq 0}$ where X_t is the active module at time t;
- when module i is active, failures times are part of a homogeneous Poisson Process (HPP) with intensity $\mu(i)$;
- when control switches from module i to module j a failure may happen with probability $\mu(i,j)$;
- when any failure appears, it does not affect the software because the execution is assumed to be restarted instantaneously.

An important issue in reliability theory, specifically for software systems, is what happens when the failure parameters tend to be smaller and smaller. Littlewood stated in (Littlewood 1975)

> *As all failure parameters $\mu(i)$,$\mu(i,j)$ tend to zero, the failure process described above is asymptotically a HPP with intensity*
> $$\lambda = \sum_i \pi(i)\big[\sum_j Q(i,j)\mu(i,j) + \mu(i)\big] \tag{1}$$

where $Q = (Q(i,j))$ and π are the generator (assumed to be irreducible) and the stationary distribution of X, respectively. This statement is well-known in the community of software reliability and has widely supported the *hierarchical approach* for modeling modular software (see e.g. (Goseva-Popstojanova and Trivedi 2001) for details). However, to the best of our knowledge, no proof of this fact is reported in the applied probability literature. The aim of this note is to provide precise statements for the asymptotic of the failure point process.

Let us denote the number of failures in interval $]0,t]$ by N_t $((N(0) = 0))$. Considering the bivariate process $Z = (N_t, X_t)_{t\geq 0}$ is very appealing from a mathematical and computational point of view. This is a jump Markov process over state space $S = \mathbb{N} \times \mathcal{M}$. $\mathcal{M}$ denotes the finite set of modules $\{1,\dots,M\}$.

The infinitesimal generator of this Markov process is

$$G = \begin{pmatrix} D_0 & D_1 & 0 & \cdots \\ 0 & D_0 & D_1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

using a lexicographic order on state space S. Matrices D_0 and D_1 are defined by

$$D_0(i,j) = \begin{cases} Q(i,j)(1-\mu(i,j)) & \text{if } i \neq j, \\ -\sum_{j\neq i} Q(i,j) - \mu(i) & \text{if } i = j, \end{cases} \qquad D_1(i,j) = \begin{cases} Q(i,j)\mu(i,j) & \text{if } i \neq j, \\ \mu(i) & \text{if } i = j. \end{cases}$$

Note that $Q = D_0 + D_1$. We refer to (Ledoux and Rubino 1997) for computational issue of various reliability metrics using the bivariate process Z.

$\mathcal{F}^N, \mathcal{F}^Z$ denote the internal histories of the counting process $N = (N_t)_{t\geq 0}$ and the bivariate Markov process $Z = (N, X)$, respectively. Due to the special structure of the generator G of Z, N may be interpreted as the following counter of specific transitions of Z

$$N_t \quad = \sum_{(x,y)\in T} \sum_{0<s\leq t} \mathbf{1}_{\{Z_{s-}=x, Z_s=y\}} = \sum_{(x,y)\in T} N_t(x,y).$$

where $T = \big\{((n,i);(n+1,j)), \quad i,j \in \mathcal{M}, n \geq 0\big\}$. It is well-known (see e.g. (Bremaud 1981)) that $(N_t(x,y) - \int_0^t \mathbf{1}_{\{Z_{s-}=x\}} G(x,y) ds)_{t\geq 0}$ is a $\mathcal{F}^Z$ martingale. Then it follows that $(N_t - A_t)_{t\geq 0}$ is a $\mathcal{F}^Z$-martingale, where

$$A_t = \int_0^t \sum_{j\in\mathcal{M}} D_1(X_{s-}, j) ds = \sum_{i\in\mathcal{M}} \big[\mu(i) + \sum_{j\neq i} Q(i,j)\mu(i,j)\big] \int_0^t \mathbf{1}_{\{X_{s-}=i\}} ds. \tag{2}$$

Process $A = (A_t)_{t\geq 0}$ is called the $\mathcal{F}^Z$-compensator of the counting process N.

2 Convergence to a Poisson process

A basic way to represent *reliability growth* is to introduce perturbated failure parameters

$$\varepsilon\mu(i), \varepsilon\mu(i,j), \quad i,j \in \mathcal{M}$$

where $\varepsilon > 0$ is a small parameter. Then we investigate convergence of the failure point process as ε tends to 0. First of all, we have to consider the counting process of the perturbated point process at time scale t/ε. That is, we investigate the convergence of the process $N^{(\varepsilon)}$ defined by $N_t^{(\varepsilon)} = N_{\frac{t}{\varepsilon}}$. Its $\mathcal{F}^{N^{(\varepsilon)},X}$-compensator is from (2)

$$A_t^{(\varepsilon)} = \sum_{i\in\mathcal{M}} \big[\mu(i) + \sum_{j\neq i} Q(i,j)\mu(i,j)\big] \varepsilon \int_0^{t/\varepsilon} \mathbf{1}_{\{X_{s-}=i\}} ds. \tag{3}$$

From the well-known time-average properties of cumulative process $\int_0^t f(X_s) ds$ for an irreducible Markov process X, we have $(\varepsilon/t) \int_0^{t/\varepsilon} \mathbf{1}_{\{X_s=i\}} ds$ converges a.s. to $\pi(i)$ where π satisfies $\pi Q = 0$. Thus, we derive that

$$A_t^{(\varepsilon)} \xrightarrow[\varepsilon\to 0]{a.s.} t \sum_{i\in\mathcal{M}} \pi(i) \big[\mu(i) + \sum_{j\neq i} Q(i,j)\mu(i,j)\big].$$

In particular, this implies the convergence in probability of $A_t^{(\varepsilon)}$ to the compensator of a HPP with intensity λ as in (1). The next theorem follows from (Kabanov, Liptser, and Shiryayev 1980, Th 1)

Theorem 1 *Probability vector π is such that $\pi Q = 0$. As ε tends to 0, the counting process $N^{(\varepsilon)} = \big(N_{t/\varepsilon}\big)$ converges in distribution to the counting process $P = (P_t)_{t\geq 0}$ of a HPP with intensity λ defined by (1)*

Note that λ in (1) is the scalar product $\langle \pi, D_1 \mathbb{1}^{\mathrm{t}}\rangle$, where $\mathbb{1}^{\mathrm{t}}$ is the M-dimensional column vector whose all entries are 1. Moreover, if $Y_s^{(\varepsilon)}$ is vector $\big(\mathbf{1}_{\{X_{s/\varepsilon}=i\}}\big)_{i\in\mathcal{M}}$, the compensator of $N^{(\varepsilon)}$ is from (3)

$$A_t^{(\varepsilon)} = \int_0^t \langle Y_{s-}^{(\varepsilon)}, D_1 \mathbb{1}^{\mathrm{t}}\rangle ds. \tag{4}$$

3 Convergence rate with distance in variation

Let T be any positive scalar. $\mathbb{T}$ is a finite subdivision $\{t_0, t_1, \dots, t_n\}$ of interval $[0,T]$ ($0 = t_0 < t_1 < \cdots < t_n = T$). To evaluate proximity between the respective distributions $\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)})$ and $\mathcal{L}(P_{\mathbb{T}})$ of $N_{\mathbb{T}}^{(\varepsilon)} = (N_{t_1}^{(\varepsilon)}, \dots, N_{t_n}^{(\varepsilon)})$ and $P_{\mathbb{T}} = (P_{t_1}, \dots, P_{t_n})$, we use their distance in total variation, denoted by $\mathrm{d_{TV}}\big(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})\big)$, that is

$$\mathrm{d_{TV}}\big(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})\big) \stackrel{\mathrm{def}}{=} \sup_{B\subset\mathbb{N}^n} \Big|\mathbb{P}\{N_{\mathbb{T}}^{(\varepsilon)} \in B\} - \mathbb{P}\{P_{\mathbb{T}} \in B\}\Big|$$

For a locally bounded variation function $t \to f(t)$, the total variation in the interval $[0,T]$ is

$$\mathrm{Var}_{[0,T]}(f) \stackrel{\mathrm{def}}{=} \sup_{\mathbb{T}\in\mathcal{P}([0,T])} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

where $\mathcal{P}([0,T])$ is the set of all the finite subdivisions of the interval $[0,T]$.

The main result is based on the following estimate (Kabanov, Liptser, and Shiryayev 1983, Th 3.1)

$$\mathrm{d_{TV}}\big(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})\big) \leq E\,\mathrm{Var}_{[0,T]}(\widehat{A}^{(\varepsilon)} - A). \tag{5}$$

where $\widehat{A}^{(\varepsilon)}$ and A are the $\mathcal{F}^{N^{(\varepsilon)}}$-compensator of $N^{(\varepsilon)}$ and the compensator of the HPP in Theorem 1, respectively. $\widehat{A}^{(\varepsilon)}$ is from (4) (Bremaud 1981)

$$\widehat{A}_t^{(\varepsilon)} = \int_0^t \langle \widehat{Y}_{s-}^{(\varepsilon)}, D_1 \mathbb{1}^{\mathrm{t}}\rangle ds$$

with $\widehat{Y}_t^{(\varepsilon)} = \big(\mathbb{P}\{X_{t/\varepsilon} = i \mid \mathcal{F}_t^{N^{(\varepsilon)}}\}\big)_{i\in\mathcal{M}}$. Therefore, we have from (5)

$$\mathrm{d_{TV}}\big(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})\big) \leq E \int_0^T |\langle \widehat{Y}_{s-}^{(\varepsilon)} - \pi, D_1 \mathbb{1}^{\mathrm{t}}\rangle| ds.$$

Note that $|\langle \widehat{Y}_s^{(\varepsilon)} - \pi, D_1 \mathbb{1}^{\mathrm{t}}\rangle| \leq \delta\, \|\widehat{Y}_s^{(\varepsilon)} - \pi\|_1$ with $\delta = \max\big((D_1\mathbb{1}^{\mathrm{t}})(i)\big) - \min\big((D_1\mathbb{1}^{\mathrm{t}})(i)\big)$ and $\|\cdot\|_1$ denotes the l_1-norm. So that, it remains to estimate the convergence rate of $\|\widehat{Y}_s^{(\varepsilon)} - \pi\|_1$ to 0 when $\varepsilon \to 0$. The first step consists in writing a filtering equation for vector $Y_t^{(\varepsilon)}$ (Bremaud 1981, Ch IV).

Lemma 1 *Define $\widehat{Y}_t^{(\varepsilon)} = E[Y_t^{(\varepsilon)} \mid \mathcal{F}_t^{N^{(\varepsilon)}}]$. We have for all $t \geq 0$*

$$\widehat{Y}_t^{(\varepsilon)} = \alpha + \frac{1}{\varepsilon}\int_0^t \widehat{Y}_{s-}^{(\varepsilon)} Q ds + \int_0^t v_{s-}^{(\varepsilon)}(dN_s^{(\varepsilon)} - \widehat{\lambda}_s ds) \tag{6}$$

where $v_{s-}^{(\varepsilon)} = \dfrac{\widehat{Y}_{s-}^{(\varepsilon)} D_1}{\widehat{\lambda}_s} - \widehat{Y}_{s-}^{(\varepsilon)}$ and $\widehat{\lambda}_s = \langle \widehat{Y}_{s-}^{(\varepsilon)}, D_1 \mathbb{1}^{\mathrm{t}}\rangle$ is the $\mathcal{F}^{N^{(\varepsilon)}}$-intensity of $N^{(\varepsilon)}$.

Equation (6) has the unique solution

$$\widehat{Y}_t^{(\varepsilon)} = \alpha \exp\big(Qt/\varepsilon\big) + \int_0^t v_{s-}^{(\varepsilon)} \exp\big(Q(t-s)/\varepsilon\big)(dN_s^{(\varepsilon)} - \widehat{\lambda}_s ds); \tag{7}$$

Theorem 2 *$P = (P_t)$ is the counting process of a HPP with intensity $\langle \pi, D_1 \mathbb{1}^{\mathrm{t}} \rangle$ where π is the probability distribution such that $\pi Q = 0$. For any $T > 0$, there exists a constant C_T such that*

$$\mathrm{d}_{\mathrm{TV}}\big(\mathcal{L}(N_{\mathbb{T}}^{(\varepsilon)}), \mathcal{L}(P_{\mathbb{T}})\big) \leq C_T\, \varepsilon.$$

Proof. Since $v_{s-}^{(\varepsilon)} \mathbb{1}^{\mathrm{t}} = 0$, we can write from (7)

$$\|\widehat{Y}_t^{(\varepsilon)} - \pi\|_1 \quad \leq \quad \|\alpha \exp\big(Qt/\varepsilon\big) - \pi\|_1 + \left\| \int_0^t v_{s-}^{(\varepsilon)} \left[\exp\big(Q(t-s)/\varepsilon\big) - \mathbb{1}^{\mathrm{t}}\pi\right] (dN_s^{(\varepsilon)} - \widehat{\lambda}_s ds)\right\|_1 .$$

Since Q is irreducible, we have the well-known exponential estimate: for all $t \geq 0$

$$\|\exp\big(Qt\big) - \mathbb{1}^{\mathrm{t}}\pi\|_1 \leq C\, \exp(-\rho t) \tag{8}$$

where ρ is positive and only depends on matrix Q.

In a first step, we get from (8)

$$\int_0^T \|\alpha \exp\big(Qt/\varepsilon\big) - \pi\|_1 dt \leq C_{1,T}\, \varepsilon.$$

In a second step, we have

$$E\left\| \int_0^t v_{s-}^{(\varepsilon)} \left[\exp\big(Q(t-s)/\varepsilon\big) - \mathbb{1}^{\mathrm{t}}\pi\right] (dN_s^{(\varepsilon)} - \widehat{\lambda}_s ds)\right\|_1 \quad \leq \quad 2E\int_0^t \left\|v_{s-}^{(\varepsilon)} \left[\exp\big(Q(t-s)/\varepsilon\big) - \mathbb{1}^{\mathrm{t}}\pi\right]\right\|_1 \widehat{\lambda}_s ds$$

since $\widehat{\lambda}_s$ is the $\mathcal{F}^{N^{(\varepsilon)}}$-intensity of $N^{(\varepsilon)}$ and $v_{s-}^{(\varepsilon)}$ is $\mathcal{F}^{N^{(\varepsilon)}}$ predictable. Note that $\|v_{s-}^{(\varepsilon)}\|_1\, \widehat{\lambda}_s$ is uniformly bounded in ε, t. Moreover, using exponential estimate (8) for $\|\exp\big(Q(t-s)/\varepsilon\big) - \mathbb{1}^{\mathrm{t}}\pi\|_1$, we derive that, for all $t \geq 0$

$$E\int_0^t \left\|v_{s-}^{(\varepsilon)} \left[\exp\big(Q(t-s)/\varepsilon\big) - \mathbb{1}^{\mathrm{t}}\pi\right]\right\|_1 \widehat{\lambda}_s ds \quad \leq \quad C_2 t\varepsilon.$$

We deduce from the previous estimate (and Fubini's theorem) that

$$\int_0^T E\left\|\int_0^t v_{s-}^{(\varepsilon)} \exp\big(Q^*(t-s)/\varepsilon - \mathbb{1}^{\mathrm{t}}\pi\big)(dN_s^{(\varepsilon)} - \widehat{\lambda}_s ds)\right\|_1 dt \quad \leq \quad C_{2,T}\, \varepsilon.$$

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